Groups

Questions

Q1.

A binary operation $\star$ on the set of non-negative integers, $\mathbb{Z}_0^+$, is defined by

$$m \star n = |m - n| \qquad m, n \in \mathbb{Z}_0^+$$

(a) Explain why $\mathbb{Z}_0^+$ is closed under the operation $\star$

(1)

(b) Show that 0 is an identity for $(\mathbb{Z}_0^+, \star)$

(2)

(c) Show that all elements of $\mathbb{Z}_0^+$ have an inverse under $\star$

(2)

(d) Determine if $\mathbb{Z}_0^+$ forms a group under $\star$, giving clear justification for your answer.

(3)

(Total for question = 8 marks)

Q2.

(i) Let G be a group of order 5 291 848

Without performing any division, use proof by contradiction to show that G cannot have a subgroup of order 11

(3)

(ii) (a) Complete the following Cayley table for the set X = 2,4,8,14,16,22,26,28 with the operation of multiplication modulo 30

$\times_{30}$	2	4	8	14	16	22	26	28
2	4	8	16	28	2	14	22	26
4	8		2			28	14	
8	16	2			8			14
14	28		22	16		8	4	
16	2	4		14	16			
22	14		26			4	2	16
26	22	14		4				8
28	26		14		28		8	

(b) Hence determine whether the set *X* with the operation of multiplication modulo 30 forms a group.

[You may assume multiplication modulo *n* is an associative operation.]

(6)

(Total for question = 9 marks)

Q3.

(i) A binary operation * is defined on positive real numbers by

$$a * b = a + b + ab$$

Prove that the operation * is associative.

(4)

(ii) The set *G* = 1, 2, 3, 4, 5, 6 forms a group under the operation of multiplication modulo 7

(a) Show that *G* is cyclic.

(2)

The set *H* = 1, 5, 7, 11, 13, 17 forms a group under the operation of multiplication modulo 18

(b) List all the subgroups of *H*.

(3)

(c) Describe an isomorphism between *G* and *H*.

(3)

(Total for question = 12 marks)

Q4.

The set e,p, q, r, s forms a group, A, under the operation *

Given that e is the identity element and that

$$p*p = s \qquad s*s = r \qquad p*p*p = q$$

(a) show that

(i) $p*q = r$
(ii) $s*p = q$

(2)

(b) Hence complete the Cayley table below.

*	*e*	*p*	*q*	*r*	*s*
e					
p					
q					
r					
s					

(2)

(c) Use your table to find $p*q*r*s$

(1)

A student states that there is a subgroup of A of order 3

(d) Comment on the validity of this statement, giving a reason for your answer.

(2)

(Total for question = 7 marks)

Q5.

The set G = 1, 3, 7, 9, 11, 13, 17, 19 under the binary operation of multiplication modulo 20 forms a group.

(a) Find the inverse of each element of G.

(3)

(b) Find the order of each element of G.

(3)

(c) Find a subgroup of G of order 4

(1)

(d) Explain how the subgroup you found in part (c) satisfies Lagrange's theorem.

(1)

(Total for question = 8 marks)

Q6.

Let G be a group of order $46^{46} + 47^{47}$

Using Fermat's Little Theorem and explaining your reasoning, determine which of the following are possible orders for a subgroup of G

(i) 11

(ii) 21

(7)

(Total for question = 7 marks)

Q7.

The group S_4 is the set of all possible permutations that can be performed on the four numbers 1, 2, 3 and 4, under the operation of composition.

For the group S_4

(a) write down the identity element,

(1)

(b) write down the inverse of the element *a*, where

$$a = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix}$$

(1)

(c) demonstrate that the operation of composition is associative using the following elements

$$a = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix} \quad b = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix} \quad \text{and } c = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}$$

(2)

(d) Explain why it is possible for the group S_4 to have a subgroup of order 4
You do not need to find such a subgroup.

(2)

(Total for question = 6 marks)

Q8.

The operation * is defined on the set S = 0, 2, 3, 4, 5, 6 by $x*y = x + y - xy$ (mod 7)

*	0	2	3	4	5	6
0						
2		0				
3						5
4						
5		4				
6						

(a) (i) Complete the Cayley table shown above

(ii) Show that S is a group under the operation *
(You may assume the associative law is satisfied.)

(6)

(b) Show that the element 4 has order 3

(2)

(c) Find an element which generates the group and express each of the elements in terms of this generator.

(3)

(Total for question = 11 marks)

Q9.

(i) A group G contains distinct elements a, b and e where e is the identity element and the group operation is multiplication.

Given $a^2b = ba$, prove $ab \neq ba$

(4)

(ii) The set H = 1, 2, 4, 7, 8, 11, 13, 14 forms a group under the operation of multiplication modulo 15

(a) Find the order of each element of H.

(3)

(b) Find three subgroups of H each of order 4, and describe each of these subgroups.

(4)

The elements of another group J are the matrices $\begin{pmatrix} \cos\left(\frac{k\pi}{4}\right) & \sin\left(\frac{k\pi}{4}\right) \\ -\sin\left(\frac{k\pi}{4}\right) & \cos\left(\frac{k\pi}{4}\right) \end{pmatrix}$
where k = 1, 2, 3, 4, 5, 6, 7, 8 and the group operation is matrix multiplication.

(c) Determine whether H and J are isomorphic, giving a reason for your answer.

(2)

(Total for question = 13 marks)

Mark Scheme - Groups

Q1.

Question	Scheme		Marks	AOs
(a)	For $m,n \in \mathbb{Z}_0^+$ we have $m-n \in \mathbb{Z}$ (difference of integers is an integer) and so $\lvert m-n \rvert \in \mathbb{Z}_0^+$, hence closed under $\star$.		B1	2.4
			(1)	
(b)	For $m \in \mathbb{Z}_0^+$, $0 \star m = \lvert 0-m \rvert = \lvert -m \rvert = m$ and $m \star 0 = \lvert m-0 \rvert = \lvert m \rvert = m$ Hence 0 is an identity*.	Checks either side	M1	1.1b
		Checks both sides and makes conclusion.	A1*	2.1
			(2)	
(c)	For $m \in \mathbb{Z}_0^+$, we need $\lvert m-n \rvert = 0 \Rightarrow n = ...$ or shows $\lvert m-m \rvert = \lvert 0 \rvert = 0$		M1	2.2a
	As $\lvert m-m \rvert = 0$ for all $m \in \mathbb{Z}_0^+$ each m is self-inverse.		A1	2.1
			(2)	
(d)	Checks associativity – ie evaluates $m \star (n \star p)$ and $(m \star n) \star p$ with letter or numbers.		M1	1.2
	E.g. $1 \star (2 \star 3) = 1 \star \lvert 2-3 \rvert = 1 \star 1 = 0$ but $(1 \star 2) \star 3 = \lvert 1-2 \rvert \star 3 = 1 \star 3 = \lvert 1-3 \rvert = 2$		M1	3.1a
	$1 \star (2 \star 3) \neq (1 \star 2) \star 3$ hence not associative so not a group.		A1	2.4
			(3)	
			(8 marks)	

Notes:

(a)

B1: Checks difference of two non-negative integers is an integer and hence its modulus is a non-negative integer and concludes closure. "Always positive" as a conclusion is B0 without consideration of the equal zero case.

(b)

M1: Checks that 0 is a left or a right identity.

A1*: Checks 0 works both sides as an identity and makes conclusion it is an identity.

(c)

M1: Realises m must be its own inverse for each m – accept if just stated m is self-inverse with no proof, or if an attempt is made to show it is self-inverse, or for an attempt to solve $\lvert m-n \rvert = 0$

A1: Each element is self-inverse with a **full** proof given.

(d)

M1: Realises associativity must be checked in some way – may be by producing a counter example, or by attempting to evaluate both sides of the associativity axiom for a general case. A statement of the correct identity is sufficient for the mark to be awarded.

M1: Produces a suitable counter example and evaluates both sides of associativity equation. Attempts at algebraic proofs are unlikely to succeed but allow the method for e.g consideration of. $m > n > p$ giving $\lvert \lvert m-n \rvert - p \rvert = \lvert m-n-p \rvert$ and $\lvert m - \lvert n-p \rvert \rvert = \lvert m-n+p \rvert$ but must have a correct reason to disambiguate the inner moduli. If in doubt use review.

A1: Must have provided a counter example. Deduces associativity does not hold and concludes $\mathbb{Z}_0^+$ is not a group under $\star$

Q2.

Question	Scheme	Marks	AOs
(i)	Suppose G has a subgroup of order 11, then (by Lagrange's Theorem) 11 must divide 5291848	M1	2.1
	But $5-2+9-1+8-4+8=23$	M1	1.1b
	23 is not divisible by 11, hence 11 does not divide $\lvert G \rvert$, which contradicts Lagrange's Theorem. Hence there is no subgroup of order 11.	A1	2.4
		(3)	

(ii)(a)

$\times_{30}$	2	4	8	14	16	22	26	28
2	4	8	16	28	2	14	22	26
4	8	**16**	2	**26**	**4**	28	14	**22**
8	16	2	**4**	**22**	8	**26**	**28**	14
14	28	**26**	22	16	**14**	8	4	**2**
16	2	4	**8**	14	16	**22**	**26**	**28**
22	14	**28**	26	**8**	**22**	4	2	16
26	22	14	**28**	4	**26**	**2**	**16**	8
28	26	**22**	14	**2**	28	**16**	8	**4**

Scheme	Marks	AOs
Completes at least one row or column correctly	M1	1.1b
At least 5 rows or columns completed correctly	A1	1.1b
Completely correct	A1	1.1b

(b)

Scheme	Marks	AOs
As the row and column for 16 repeat the borders, 16 is an identity element for $(X, \times_{30})$	B1	2.2a
Each element has an inverse as follows: (see table below)	B1	1.1b

x	2	4	8	14	16	22	26	28
x^{-1}	8	4	2	14	16	28	26	22

Scheme	Marks	AOs
Since we know $\times_{30}$ is associative and as there are no new elements in the table, so $(X, \times_{30})$ is closed, hence $(X, \times_{30})$ is a group.	B1	2.4
	(6)	

(9 marks)

Notes:
(i) **M1:** Sets up the proof by stating or implying that if there is a subgroup of order 11 then by Lagrange's Theorem 11 must divide 5291848. May not mention Lagrange's Theorem at this stage. A formal assumption is not required as long as it is implicit. **M1:** Applies the divisibility test for 11. Look for an attempt at the alternating sum being used. **A1:** Alternating sum is 23, so derives a contradiction as 11 does not divide $\|G\|$, and conclusion made. Use of Lagrange's Theorem must be clear, though it need not be named. (ii)(a) **M1:** Begins process of completing the table by filling in at least one row or column correctly. **A1:** Five or more rows or columns completed correctly. **A1:** Completely correct table. (b) **B1:** Identifies 16 as the identity element. No reason needed. **B1:** Identifies all inverses or gives reason why each element has an inverse (may refer to each row and column containing the identity once only and symmetrically about the diagonal). **B1:** Refers to closure and associativity to deduce $(X, \times_{30})$ is a group. SC Allow B0B0B1ft for deducing not a group with valid reason if identity or inverse checks fail.

Q3.

Question	Scheme	Marks	AOs
(i)	$(a*b)*c=(a+b+ab)*c=a+b+ab+c+(a+b+ab)c$	M1	2.1
	$a*(b*c)=a*(b+c+bc)=a+b+c+bc+a(b+c+bc)$	M1	2.1
	$\underline{a+b+ab+c+(a+b+ab)c}=a+b+c+bc+ab+ac+abc$ $=\underline{a+b+c+bc+a(b+c+bc)}$	A1	2.2a
	so $(a*b)*c=a*(b*c)$ which means * is associative	A1	2.4
		(4)	
(ii)(a)	$3^2=2\ \ 3^3=6\ \ 3^4=4\ \ 3^5=5\ \ 3^6=1$ or $5^2=4\ \ 5^3=6\ \ 5^4=2\ \ 5^5=3\ \ 5^6=1$	M1	2.1
	Or special case for M1A0 if powers not shown: 3 has order 6 so generates the group		
	3 (or 5) has order 6 and so generates the group so G is cyclic	A1	2.4
		(2)	
(b)	$\{1\}$, H	B1	1.1b
	$\{1, 17\}$ **or** $\{1, 7, 13\}$	M1	1.1b
	$\{1, 17\}$ **and** $\{1, 7, 13\}$ (and no others)	A1	1.1b
		(3)	
(c)	G: 1, 2, 3, 4, 5, 6 / H: 1, 7, 5, 13, 11, 17 **or** G: 1, 2, 3, 4, 5, 6 / H: 1, 13, 11, 7, 5, 17	M1 A1 A1	3.1a 1.1b 1.1b
		(3)	
		(12 marks)	

Notes
(i)
M1: Begins proof by correctly expanding $(a*b)*c$ **or** $a*(b*c)$ to an expression in a, b and c. Note they may expand as $(a*b)*c=(a*b)+c+(a*b)c=a+b+ab+c+(a+b+ab)c$ which is equally fine.
M1: Makes progress towards the required result by attempting to expand both $(a*b)*c$ **and** $a*(b*c)$, but be generous with the attempts for this method. May achieve this by working from left to right, so look for arriving at the other expression through a chain of equalities.
A1: For both underlined expressions (but accept eg. $c(a + b + ab)$ for $(a + b + ab)c$) and a correct expansion seen for each independently or part of a chain as shown. The expansion may have terms in different orders.
A1: Explains that $(a*b)*c=a*(b*c)$ means that * is associative. Depends on both M marks and a correct expression having been found.

(ii)(a)
M1: Demonstrates understanding of the term cyclic by either attempting all the powers of 3 or 5. Accept for this a statement $\langle 3\rangle = \{3, 2, 6, 4, 5, 1\}$ which shows the elements list in order of powers.
A1: Must have evaluated all powers of 3 or 5 correctly and explains why the group is cyclic. Accept as 3 generates the group, or as 3 has the same order of *G* as reason. Must refer to cyclic in conclusion.
Special case: Allow M1A0 for a correct explanation of why *G* is cyclic if the order of 3 (or 5) is stated as 6 without justification – but must include reference to either being a generator or having the same order as *G*.
(b) (You may ignore references to the operation for this part)
B1: Identifies $\{1\}$ and *H* as subgroups
M1: Identifies $\{1, 17\}$ **or** $\{1, 7, 13\}$ as a subgroup
A1: Identifies $\{1, 17\}$ **and** $\{1, 7, 13\}$ as subgroups and no others
(c)
M1: Attempts to identify an isomorphism between the groups – may be implied by

- identifying at least 2 correct non-identity pairings or
- by attempting to rearrange group tables to have the same structure, or
- by attempting to map powers of a generator to powers of a generator e.g $(\text{their } 3)^k \to (\text{their } 5)^k$ or
- by matching of non-trivial proper subgroups to each other.

A1: Identifies 4 correct pairings, or sets up a mapping with one correct generator
A1: All pairings correct, or sets up a mapping with generators of each group correct, eg. $3^k \to 5^k$

Q4.

Question	Scheme	Marks	AOs
(a)	$p*q = p*p*p*p = s*s = r$ OR $s*s = r \Rightarrow p*p*p*p = r \Rightarrow p*q = r$	B1	2.1
	$s*p = p*p*p = q$ OR as $p*p*p = q$ and $p*p = s \Rightarrow s*p = q$	B1	2.1
		(2)	
(b)	see table below	M1 A1	1.1b 1.1b
		(2)	
(c)	$p*q*r*s = e$	B1	1.1b
		(1)	
(d)	The order of a subgroup is a factor of the order of the group (Lagrange's Theorem)	M1	1.2
	As 3 is not a factor of 5, the student's statement is wrong	A1	2.3
		(2)	
		(7 marks)	

Table for (b):

*	**e**	**p**	**q**	**r**	**s**
e	e	p	q	r	s
p	p	s	r	e	q
q	q	r	p	s	e
r	r	e	s	q	p
s	s	q	e	p	r

Notes
(a) **B1:** Correct proof to achieve the printed statement **B1:** Correct proof to achieve the printed statement
(b) Marked B1 B1 on ePen **M1:** Finds at least 13 correct entries – usually the highlighted **A1:** Completely correct table
(c) **B1:** See scheme
(d) **M1:** Some indication that the order of a subgroup must be a factor of the order of the group. May say that 3 is not a factor of 5 or equivalent **A1:** Fully correct unambiguous statement that refers Lagrange's theorem and either • 3 is not a factor of 5 • 3 does not divide 5 • 5 is not divisible by 3 and comments that the student's statement is incorrect. No contradictory statements

Q5.

<table>
<tr><th>Question</th><th>Scheme</th><th>Marks</th><th>AOs</th></tr>
<tr><td>(a)</td><td>1, 9, 11 and 19 are self-inverse</td><td>M1
A1</td><td>1.1b
1.1b</td></tr>
<tr><td></td><td><table><tr><td>3</td><td>7</td><td>13</td><td>17</td></tr><tr><td>7</td><td>3</td><td>17</td><td>13</td></tr></table></td><td>B1</td><td>1.1b</td></tr>
<tr><td></td><td></td><td>(3)</td><td></td></tr>
<tr><td>(b)</td><td><table><tr><td>1</td><td>3</td><td>7</td><td>9</td><td>11</td><td>13</td><td>17</td><td>19</td></tr><tr><td>1</td><td>4</td><td>4</td><td>2</td><td>2</td><td>4</td><td>4</td><td>2</td></tr></table></td><td>M1
A1
A1</td><td>1.1b
1.1b
1.1b</td></tr>
<tr><td></td><td></td><td>(3)</td><td></td></tr>
<tr><td>(c)</td><td>$\{1, 3, 7, 9\}$ or $\{1, 9, 13, 17\}$ or $\{1, 9, 11, 19\}$</td><td>B1</td><td>2.5</td></tr>
<tr><td></td><td></td><td>(1)</td><td></td></tr>
<tr><td>(d)</td><td>Because 4 is a factor of 8</td><td>B1</td><td>2.4</td></tr>
<tr><td></td><td></td><td>(1)</td><td></td></tr>
<tr><td colspan="4" align="right">(8 marks)</td></tr>
<tr><th colspan="4">Notes</th></tr>
<tr><td colspan="4">(a)
M1: For any 2 of the self-inverse elements
A1: All 4 self-inverse elements correctly identified
B1: Correct inverses for the other elements
(b)
M1: At least 3 correct orders
A1: 6 correct orders
A1: All correct
(c)
B1: Describes a correct subgroup of order 4
(d)
B1: Correct explanation</td></tr>
</table>

Q6.

Question	Scheme	Marks	AOs
(i)	(Order of a subgroup must divide the order of a group by Lagrange's Theorem), so need to check if 11 (and/or 21) divides $46^{46}+47^{47}$ and by FLT, e.g. $a^{11-1}=a^{10}\equiv 1(\bmod 11)$, so	M1	1.1b
	$46^{46}+47^{47}\equiv 2^{4\times10+6}+3^{4\times10+7}\equiv 2^6+3^7\equiv 64+(3^3)^2\times 3$ $\equiv 9+5^2\times 3\equiv 84\equiv 7(\bmod 11)$	M1	3.1a
	Hence 11 is not a divisor of $46^{46}+47^{47}$ so not a possible order for a subgroup.	A1	2.2a
(ii)	$21=7\times 3$ so need to check for factors of 7 and 3, using $a^2\equiv 1(\bmod 3)$ and $a^6\equiv 1(\bmod 7)$	M1	3.1a
	$46^{46}+47^{47}\equiv 1^{46}+2^{47}\equiv 1+2^{2\times 23+1}\equiv 1+2^1\equiv 3\equiv 0(\bmod 3)$	M1	1.1b
	$46^{46}+47^{47}\equiv 4^{46}+(-2)^{47}\equiv 4^{6\times7+4}+(-2)^{6\times7+5}\equiv 4^4+(-2)^5$ $\equiv 16^2-32\equiv 9^2-4\equiv 81-4\equiv 77\equiv 0(\bmod 7)$	M1	2.1
	As $46^{46}+47^{47}$ divisible by both 3 and 7 it is divisible by 21 and hence this is a possible order for a subgroup.	A1	2.4
		(7)	
			(7 marks)

Notes:
(i) **M1**: For an attempt to apply a correct Fermat's Little theorem at least once in the question with either $p=11$, $p=7$ or $p=3$ on either the 46^{46} or 47^{47} term. **M1**: Applies FLT and congruence arithmetic fully to find the residue of $46^{46}+47^{47}$ modulo 11. There will be lots of different routes, so look for an attempt to apply FLT that leads to determining if 11 is a divisor or not. **A1**: $46^{46}+47^{47}\equiv 7(\bmod 11)$ (accept equivalents as long as it is clear it is not congruent to 0) and deduces it is not a possible order for a subgroup.
(ii) **M1**: Applies checks for both 7 and 3 as divisors of $46^{46}+47^{47}$ via similar strategy. **M1**: Applies FLT with $p=3$ to find a smaller residue modulo 3. Other routes are possible. **M1**: Applies FLT with $p=7$ to find a smaller residue modulo 7. Other routes are possible. **A1**: Shows $46^{46}+47^{47}$ congruent to 0 modulo 3 and modulo7, and deduces 21 divides $46^{46}+47^{47}$ hence it is a possible order for a subgroup. Alt: **M1**: Reduces the bases modulo 21 and applies a power reduction technique using congruences for at least one of the power of 46 or 47 **M1**: Reduces fully by congruence arithmetic either the 46^{46} or 47^{47} term. **M1**: Reduces fully by congruence arithmetic both the 46^{46} and 47^{47} terms **A1**: Shows $46^{46}+47^{47}$ congruent to 0 modulo 21, and deduces 21 divides $46^{46}+47^{47}$ hence it is a possible order for a subgroup.

Q7.

Question	Scheme	Marks	AOs
(a)	$\{e =\}\begin{pmatrix}1 & 2 & 3 & 4\\1 & 2 & 3 & 4\end{pmatrix}$	B1	1.1b
		(1)	
(b)	$\begin{pmatrix}1 & 2 & 3 & 4\\4 & 3 & 1 & 2\end{pmatrix}$	B1	1.1b
		(1)	
(c)	Demonstrates that, for example: $[a\circ b]\circ c = \left[\begin{pmatrix}1 & 2 & 3 & 4\\3 & 4 & 2 & 1\end{pmatrix}\circ\begin{pmatrix}1 & 2 & 3 & 4\\2 & 4 & 3 & 1\end{pmatrix}\right]\circ\begin{pmatrix}1 & 2 & 3 & 4\\4 & 1 & 2 & 3\end{pmatrix}$ $= \begin{pmatrix}1 & 2 & 3 & 4\\4 & 1 & 2 & 3\end{pmatrix}\circ\begin{pmatrix}1 & 2 & 3 & 4\\4 & 1 & 2 & 3\end{pmatrix} = \begin{pmatrix}1 & 2 & 3 & 4\\3 & 4 & 1 & 2\end{pmatrix}$ $a\circ[b\circ c] = \begin{pmatrix}1 & 2 & 3 & 4\\3 & 4 & 2 & 1\end{pmatrix}\circ\left[\begin{pmatrix}1 & 2 & 3 & 4\\2 & 4 & 3 & 1\end{pmatrix}\circ\begin{pmatrix}1 & 2 & 3 & 4\\4 & 1 & 2 & 3\end{pmatrix}\right]$ $= \begin{pmatrix}1 & 2 & 3 & 4\\3 & 4 & 2 & 1\end{pmatrix}\circ\begin{pmatrix}1 & 2 & 3 & 4\\1 & 2 & 4 & 3\end{pmatrix} = \begin{pmatrix}1 & 2 & 3 & 4\\3 & 4 & 1 & 2\end{pmatrix}$	M1	2.1
	So $[a\circ b]\circ c = a\circ[b\circ c]$ or associative	A1	2.4
		(2)	
(d)	The order of the group is 24 or 4!	B1	1.1b
	4 is a factor of 24 or **4/24** therefore **it is possible** for a subgroup to have order 4.	B1ft	2.4
		(2)	
		(6 marks)	

Notes:
(a) **B1**: See scheme
(b) **B1**: See scheme
(c) **M1**: Shows two calculations in an attempt to show associative, e,g, $[a\circ b]\circ c$ and $a\circ[b\circ c]$. There must be an intermediate line of working with evidence of using the permutations. Condone the wrong order for this mark. **A1**: Correct calculations leading to $[a\circ b]\circ c = a\circ[b\circ c]$ or states associative **Note Incorrect order scores M1 A0** $[a\circ b]\circ c = \left[\begin{pmatrix}1 & 2 & 3 & 4\\3 & 4 & 2 & 1\end{pmatrix}\circ\begin{pmatrix}1 & 2 & 3 & 4\\2 & 4 & 3 & 1\end{pmatrix}\right]\circ\begin{pmatrix}1 & 2 & 3 & 4\\4 & 1 & 2 & 3\end{pmatrix}$ $= \begin{pmatrix}1 & 2 & 3 & 4\\3 & 1 & 4 & 2\end{pmatrix}\circ\begin{pmatrix}1 & 2 & 3 & 4\\4 & 1 & 2 & 3\end{pmatrix} = \begin{pmatrix}1 & 2 & 3 & 4\\2 & 4 & 3 & 1\end{pmatrix}$ $= \begin{pmatrix}1 & 2 & 3 & 4\\3 & 4 & 2 & 1\end{pmatrix}\circ\begin{pmatrix}1 & 2 & 3 & 4\\1 & 3 & 2 & 4\end{pmatrix} = \begin{pmatrix}1 & 2 & 3 & 4\\2 & 4 & 3 & 1\end{pmatrix}$
(d) **B1**: Order is 24 or 4! **B1ft**: Follow through on their order of the group, draws the correct conclusion

Q8.

<table>
<tr><th>Question</th><th>Scheme</th><th>Marks</th><th>AOs</th></tr>
<tr><td>(a)</td><td>(i)
<table>
<tr><td>*</td><td>0</td><td>2</td><td>3</td><td>4</td><td>5</td><td>6</td></tr>
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<tr><td>4</td><td>4</td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td>5</td><td>5</td><td>4</td><td></td><td></td><td></td><td></td></tr>
<tr><td>6</td><td>6</td><td></td><td>5</td><td></td><td></td><td></td></tr>
</table>
</td><td>M1</td><td>1.1b</td></tr>
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</table>
</td><td>M1 A1</td><td>1.1b
1.1b</td></tr>
<tr><td></td><td>(ii) Identity is zero and there is closure as shown above</td><td>M1</td><td>2.1</td></tr>
<tr><td></td><td>3 and 5 are inverses, 4 and 6 are inverses, 2 is self inverse,
0 is identity so is self inverse</td><td>M1</td><td>2.5</td></tr>
<tr><td></td><td>Associative law may be assumed so S forms a group</td><td>A1</td><td>1.1b</td></tr>
<tr><td></td><td></td><td>(6)</td><td></td></tr>
</table>

(b)	4*4*4 = 4* (4 * 4) = 4 * 6 or 4*4*4 = (4* 4) * 4 = 6* 4	M1	2.1
	= 0 (the identity) so 4 has order 3	A1	2.2a
		(2)	
(c)	3 and 5 each have order 6 so either generates the group	M1	3.1a
	Either $3^1 = 3,\ 3^2 = 4,\ 3^3 = 2,\ 3^4 = 6,\ 3^5 = 5,\ 3^6 = 0$ **Or** $5^1 = 5,\ 5^2 = 6,\ 5^3 = 2,\ 5^4 = 4,\ 5^5 = 3,\ 5^6 = 0$	A1, A1	1.1b 1.1b
		(3)	
			(11 marks)

Notes:

(a)(i)

M1: Begins completing the table – obtaining correct first row and first column and using symmetry
M1: Mostly correct – three rows or three columns correct (so demonstrates understanding of using *
A1: Completely correct

(a)(ii)

M1: States closure and identifies the identity as zero
M1: Finds inverses for each element

A1: States that associative law is satisfied and so all axioms satisfied and S is a group
(b) **M1:** Clearly begins process to find 4*4*4 reaching 6*4 or 4*6 with clear explanation **A1:** Gives answer as zero, states identity and deduces that order is 3
(c) **M1:** Finds either 3 or 5 or both **A1:** Expresses four of the six terms as powers of either generator correctly (may omit identity and generator itself) **A1:** Expresses all six terms correctly in terms of either 3 or 5 (Do not need to give both)

Q9.

Question	Scheme	Marks	AOs
(i)	If we assume $ab = ba$; as $a^2b = ba$ then $ab = a^2b$	M1	2.1
	So $a^{-1}abb^{-1} = a^{-1}a^2bb^{-1}$	M1	2.1
	So $e = a$	A1	2.2a
	But this is a contradiction, as the elements e and a are distinct so $ab \neq ba$	A1	2.4
		(4)	
(ii)(a)	2 has order 4 and 4 has order 2	M1	1.1b
	7, 8 and 13 have order 4	A1	1.1b
	11 and 14 have order 2 and 1 has order 1	A1	1.1b
		(3)	
(ii)(b)	Finds the subgroup {1, 2, 4, 8} or the subgroup {1, 7, 4, 13}	M1	1.1b
	Finds both and refers to them as cyclic groups, or gives generator 2 and generator 7	A1	2.4
	Finds {1, 4, 11, 14}	B1	2.2a
	States each element has order 2 or refers to it as Klein Group	B1	2.5
		(4)	
(ii)(c)	J has an element of order 8, (H does not) or J is a cyclic group (H is not) or other valid reason	M1	2.4
	They are not isomorphic	A1	2.2a
		(2)	
		(13 marks)	

Notes:	
(i)	
M1:	Proof begins with assumption that $ab = ba$ and deduces that this implies $ab = a^2b$
M1:	A correct proof with working shown follows, and may be done in two stages
A1:	Concludes that assumption implies that $e = a$
A1:	Explains clearly that this is a contradiction, as the elements e and a are distinct so $ab \neq ba$
(ii)(a)	
M1:	Obtains two correct orders (usually the two in the scheme)
A1:	Finds another three correctly
A1:	Finds the final three so that all eight are correct
(ii)(b)	
M1:	Finds one of the cyclic subgroups
A1:	Finds both subgroups and explains that they are cyclic groups, or gives generators 2 and 7
B1:	Finds the non cyclic group
B1:	Uses correct terms that each element has order 2 or refers to it as Klein Group
(ii)(c)	
M1:	Clearly explains how J differs from H
A1:	Correct deduction